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## Note on representations of generalized inverse $*$ -semigroups<sup>1</sup>

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### Abstract

The Munn representation of an inverse semigroup  $S$ , in which the semigroup is represented by isomorphisms between principal ideals of the semilattice  $E(S)$ , is not always faithful. By introducing a concept of a *presemilattice*, Reilly considered of enlarging the carrier set  $E(S)$  of the Munn representation in order to obtain a faithful representation of  $S$  as an inverse subsemigroup of a structure resembling the Munn semigroup  $T_{E(S)}$ .

The purpose of this paper is to obtain a generalization of the Reilly's results for generalized inverse  $*$ -semigroups.

## 1 Introduction

A semigroup  $S$  with a unary operation  $*$  :  $S \rightarrow S$  is called a *regular  $*$ -semigroup* if it satisfies

- (i)  $(x^*)^* = x$ ,
- (ii)  $(xy)^* = y^*x^*$ ,
- (iii)  $xx^*x = x$ .

Let  $S$  be a regular  $*$ -semigroup. An idempotent  $e$  in  $S$  is called a *projection* if it satisfies  $e^* = e$ . For any subset  $A$  of  $S$ , denote the sets of idempotents and projections of  $A$  by  $E(A)$  and  $P(A)$ , respectively.

Let  $S$  be a regular  $*$ -semigroup. It is called a *locally inverse  $*$ -semigroup* if, for any  $e \in E(S)$ ,  $eSe$  is an inverse subsemigroup of  $S$ . If  $E(S)$  is a normal band, then  $S$  is called a *generalized inverse  $*$ -semigroup*.

Let  $S$  and  $T$  be regular  $*$ -semigroups. A homomorphism  $\phi : S \rightarrow T$  is called a  *$*$ -homomorphism* if  $(a\phi)^* = a^*\phi$ . A congruence  $\sigma$  on  $S$  is called a  *$*$ -congruence* if

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<sup>1</sup>This is the abstract and the details will be published elsewhere

$(a\sigma)^* = a^*\sigma$ . A  $*$ -congruence  $\sigma$  on  $S$  is said to be *idempotent-separating* if  $\sigma \subseteq \mathcal{H}$ , where  $\mathcal{H}$  is one of the Green's relations. Denote the maximum idempotent-separating  $*$ -congruence on  $S$  by  $\mu_S$  or simply by  $\mu$ . If  $\mu_S$  is the identity relation on  $S$ ,  $S$  is called *fundamental*. The following results are well-known, and we use them frequently throughout this paper.

**Result 1.1** [2]. *Let  $S$  be a regular  $*$ -semigroup. Then we have the following:*

- (1)  $E(S) = P(S)^2$ ;
- (2) for any  $a \in S$  and  $e \in P(S)$ ,  $a^*ea \in P(S)$ ;
- (3) each  $\mathcal{L}$ -class and each  $\mathcal{R}$ -class have one and only one projection;
- (4)  $\mu_S = \{(a, b) \in S \times S : a^*ea = b^*eb \text{ and } aea^* = beb^* \text{ for all } e \in P(S)\}$ .

For a mapping  $\alpha : A \rightarrow B$ , denote the domain and the range of  $\alpha$  by  $d(\alpha)$  and  $r(\alpha)$ , respectively. For a subset  $C$  of  $A$ ,  $\alpha|_C$  means the restriction of  $\alpha$  to  $C$ .

As a generalization of the Preston-Vagner representations, one of the authors gave two types of representations of locally [generalized] inverse  $*$ -semigroups in [3], [4] and [5]. In this paper, we follow [5]. A non-empty set  $X$  with a reflexive and symmetric relation  $\sigma$  is called an  $\iota$ -set, and denoted by  $(X; \sigma)$ . If  $\sigma$  is transitive, that is, if  $\sigma$  is an equivalence relation on  $X$ ,  $(X; \sigma)$  is called a *transitive  $\iota$ -set*.

Let  $(X; \sigma)$  be an  $\iota$ -set. A subset  $A$  of  $X$  is called an  $\iota$ -single subset of  $(X; \sigma)$  if it satisfies the following condition:

for any  $x \in X$ , there exists at most one element  $y \in A$  such that  $(x, y) \in \sigma$ .

We consider the empty set to be an  $\iota$ -single subset. We remark that if  $(X; \sigma)$  is a transitive  $\iota$ -set, a subset  $A$  of  $X$  is an  $\iota$ -single subset if and only if, for  $x, y \in A$ ,  $(x, y) \in \sigma$  implies  $x = y$ . A mapping  $\alpha$  in  $\mathcal{I}_X$ , the symmetric inverse semigroup on  $X$ , is called a *partial one-to-one  $\iota$ -mapping* on  $(X; \sigma)$  if  $d(\alpha), r(\alpha)$  are both  $\iota$ -single subsets of  $(X; \sigma)$ , where  $d(\alpha)$  and  $r(\alpha)$  are the domain and the range of  $\alpha$ , respectively. Denote the set of all partial one-to-one  $\iota$ -mappings of  $(X; \sigma)$  by  $\mathcal{LI}_{(X; \sigma)}$ .

For any  $\iota$ -single subsets  $A$  and  $B$  of  $(X; \sigma)$ , define  $\theta_{A, B}$  by

$$\theta_{A, B} = \{(a, b) \in A \times B : (a, b) \in \sigma\} = (A \times B) \cap \sigma.$$

Since a subset of an  $\iota$ -single subset is also an  $\iota$ -single subset,  $\theta_{A, B} \in \mathcal{LI}_{(X; \sigma)}$ . For any  $\alpha, \beta \in \mathcal{LI}_{(X; \sigma)}$ , define  $\theta_{\alpha, \beta}$  by  $\theta_{\alpha, \beta} = \theta_{r(\alpha), d(\beta)}$ , and let  $\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in \mathcal{LI}_{(X; \sigma)}\}$ , an indexed set of one-to-one partial functions. Now, define a multiplication  $\circ$  and a unary operation  $*$  on  $\mathcal{LI}_{(X; \sigma)}$  as follows:

$$\alpha \circ \beta = \alpha \theta_{\alpha, \beta} \beta \quad \text{and} \quad \alpha^* = \alpha^{-1},$$

where the multiplication of the right side of the first equality is that of  $\mathcal{I}_X$ . Denote  $(\mathcal{LI}_{(X;\sigma)}, \circ, *)$  by  $\mathcal{LI}_{(X;\sigma)}(\mathcal{M})$  or simply by  $\mathcal{LI}_{(X;\sigma)}$ . In this paper, we use  $\mathcal{LI}_{(X;\sigma)}$  rather than  $\mathcal{LI}_{(X;\sigma)}(\mathcal{M})$ .

**Result 1.2** [5]. *For any  $\iota$ -set  $(X; \sigma)$ ,  $\mathcal{LI}_{(X;\sigma)}$ , defined above, is a locally inverse  $*$ -semigroup. If  $(X; \sigma)$  is a transitive  $\iota$ -set, then  $\mathcal{LI}_{(X;\sigma)}$  is a generalized inverse  $*$ -semigroup. In this case, we denote it by  $\mathcal{GI}_{(X;\sigma)}$  instead of  $\mathcal{LI}_{(X;\sigma)}$ .*

*Moreover, if  $\sigma$  is the identity relation on  $X$ , then  $\mathcal{LI}_{(X;\sigma)}$  is the symmetric inverse semigroup  $\mathcal{I}_X$  on  $X$ .*

We call  $\mathcal{LI}_{(X;\sigma)}$  [ $\mathcal{GI}_{(X;\sigma)}$ ] the  $\iota$ -symmetric locally [generalized] inverse  $*$ -semigroup on the  $\iota$ -set [the transitive  $\iota$ -set]  $(X; \sigma)$  with the structure sandwich set  $\mathcal{M}$ .

Let  $S$  be a regular  $*$ -semigroup, and define a relation  $\Omega$  on  $S$  as follows:

$$(x, y) \in \Omega \iff \text{there exists } e \in E(S) \text{ such that } x\rho_e = y,$$

where  $\rho_a (a \in S)$  is the mapping of  $Sa^*$  onto  $Sa$  defined by  $x\rho_a = xa$ .

**Result 1.3** [5]. *Let  $S$  be a locally inverse  $*$ -semigroup. For each  $a \in S$ , let*

$$\rho_a : x \mapsto xa \quad (x \in d(\rho_a) = Sa^*).$$

*Then a mapping*

$$\rho : a \mapsto \rho_a$$

*is a  $*$ -monomorphism of  $S$  into  $\mathcal{LI}_{(S;\Omega)}(\mathcal{M})$ .*

For a partial groupoid  $X$ , if there exist a semilattice  $Y$ , a partition  $\pi : X \sim \sum \{X_e : e \in Y\}$  of  $X$  and mappings  $\varphi_{e,f} : X_e \rightarrow X_f$  ( $e \geq f$  in  $Y$ ) such that

- (1) for any  $e \in Y$ ,  $\varphi_{e,e} = 1_{X_e}$ ,
- (2) if  $e \geq f \geq g$ , then  $\varphi_{e,f}\varphi_{f,g} = \varphi_{e,g}$ ,
- (3) for  $x \in X_e$ ,  $y \in X_f$ ,  $xy$  is defined in  $X$  if and only if  $x\varphi_{e,ef} = y\varphi_{f,ef}$ , and in this case  $xy = x\varphi_{e,ef}$ ,

then  $X$  is called a *strong  $\pi$ -groupoid* with mappings  $\{\varphi_{e,f} : e, f \in Y, e \geq f\}$ , and it is denoted by  $X(\pi; Y; \{\varphi_{e,f}\})$  or simply by  $X(\pi)$ .

Let  $X(\pi; Y; \{\varphi_{e,f}\})$  be a strong  $\pi$ -groupoid. A subset  $A$  of  $X$  is called a  *$\pi$ -singleton subset* of  $X(\pi; Y; \{\varphi_{e,f}\})$ , if there exists  $e \in Y$  such that

$$|A \cap X_f| = \begin{cases} 1 & \text{if } f \in \langle e \rangle, \\ 0 & \text{otherwise,} \end{cases}$$

$$(A \cap X_f)\varphi_{f,g} = A \cap X_g \quad \text{for any } f, g \in \langle e \rangle \text{ such that } f \geq g,$$

where  $\langle e \rangle$  is the principal ideal of  $Y$  generated by  $e$ . In this case, we sometimes denote the  $\pi$ -singleton subset  $A$  by  $A(e)$ . If  $A(e)$  is a  $\pi$ -singleton subset, then  $|A \cap X_f| = 1$  for any  $f \in \langle e \rangle$ . We denote the only one element of  $A \cap X_f$  by  $a_f$ . We remark that, for any  $\pi$ -singleton subset  $A(e)$ ,  $A(e) = \{a_e \varphi_{e,f} : f \in \langle e \rangle\}$ . Denote the set of all  $\pi$ -singleton subsets of  $X(\pi; Y; \{\varphi_{e,f}\})$  by  $\mathcal{X}$ .

Two  $\pi$ -singleton subsets  $A(e)$  and  $B(f)$  are said to be  $\pi$ -isomorphic to each other, if there exists an isomorphism  $\bar{\alpha} : \langle e \rangle \rightarrow \langle f \rangle$  as semilattices. In this case, the mapping  $\alpha : A(e) \rightarrow B(f)$  defined by  $a_g \alpha = b_{g\bar{\alpha}}$  ( $g \in \langle e \rangle$ ) is called a  $\pi$ -isomorphism of  $A(e)$  to  $B(f)$ . It is obvious that  $\alpha$  is a bijection of  $A(e)$  onto  $B(f)$ , and hence  $\alpha \in \mathcal{I}_X$ .

Let  $X(\pi; Y; \{\varphi_{e,f}\})$  be a strong  $\pi$ -groupoid. Define an equivalence relation  $\mathcal{U}$  on  $\mathcal{X}$  by

$$\mathcal{U} = \{(A(e), B(f)) \in \mathcal{X} \times \mathcal{X} : \langle e \rangle \cong \langle f \rangle \text{ (as semilattices)}\}.$$

For  $(A(e), B(f)) \in \mathcal{U}$ , let  $T_{A(e), B(f)}$  be the set of all  $\pi$ -isomorphisms of  $A(e)$  onto  $B(f)$ , and let

$$T_{X(\pi)} = \bigcup_{(A(e), B(f)) \in \mathcal{U}} T_{A(e), B(f)}.$$

For any  $\alpha, \beta \in T_{X(\pi)}$ , define a mapping  $\theta_{\alpha, \beta}$  as follows:

$$\begin{aligned} d(\theta_{\alpha, \beta}) &= \{a \in r(\alpha) : \text{there exist } e \in Y \text{ and } b \in d(\beta) \text{ such that } a, b \in X_e\}, \\ r(\theta_{\alpha, \beta}) &= \{b \in d(\beta) : \text{there exist } e \in Y \text{ and } a \in r(\alpha) \text{ such that } a, b \in X_e\}, \\ a\theta_{\alpha, \beta} &= b \quad \text{if } r(\alpha) \cap X_e = \{a\} \text{ and } d(\beta) \cap X_e = \{b\}. \end{aligned}$$

Then  $\theta_{\alpha, \beta} \in T_{X(\pi)}$ . Let  $\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in T_{X(\pi)}\}$ , and define a multiplication  $\circ$  and a unary operation  $*$  on  $T_{X(\pi)}$  by

$$\begin{aligned} \alpha \circ \beta &= \alpha \theta_{\alpha, \beta} \beta, \\ \alpha^* &= \alpha^{-1}. \end{aligned}$$

Then  $T_{X(\pi)}(\circ, *)$  is a regular  $*$ -semigroup. We denote it by  $T_{X(\pi)}(\mathcal{M})$ .

**Result 1.4** [4]. *A regular  $\ast$ -semigroup  $T_{X(\pi)}(\mathcal{M})$  is a generalized inverse  $\ast$ -semigroup whose set of projections is partially isomorphic to  $X$ .*

Let  $S$  be a generalized inverse  $\ast$ -semigroup. Hereafter, denote  $E(S)$  and  $P(S)$  simply by  $E$  and  $P$ , respectively. Let  $E \sim \sum\{E_i : i \in I\}$  be the structure decomposition of  $E$ , and let  $P_i = P(E_i)$ . Then  $\pi : P \sim \sum\{P_i : i \in I\}$  is a partition of  $P$ . For any  $i, j \in I$  ( $i \geq j$ ), define a mapping  $\varphi_{i,j} : P_i \rightarrow P_j$  by

$$e\varphi_{i,j} = efe \quad \text{for some (any) } f \in P_j.$$

Then  $P(\pi; I; \{\varphi_{i,j}\})$  is a strong  $\pi$ -groupoid.

**Result 1.5** [4]. *Let  $S$  be a generalized inverse  $\ast$ -semigroup. For each  $a \in S$ , let*

$$\tau_a : e \mapsto a^\ast ea \quad (e \in d(\tau_a) = P(Sa^\ast)).$$

*Then a mapping  $\tau : a \mapsto \tau_a$  is a  $\ast$ -homomorphism of  $S$  into  $T_{P(\pi)}(\mathcal{M})$  such that  $\tau \circ \tau^{-1} = \mu$ .*

A regular  $\ast$ -subsemigroup  $T$  of a regular  $\ast$ -semigroup  $S$  is said to be  $\mathcal{P}$ -full if  $P(T) = P(S)$ .

**Result 1.6** [4]. *A generalized inverse  $\ast$ -semigroup  $S$  is fundamental if and only if it is  $\ast$ -isomorphic to a  $\mathcal{P}$ -full generalized inverse  $\ast$ -subsemigroup of  $T_{X(\pi)}(\mathcal{M})$  on a strong  $\pi$ -groupoid  $X(\pi; I; \{\varphi_{i,j}\})$  such that  $P(T_{X(\pi)}(\mathcal{M}))$  is partially isomorphic to  $P(S)$ .*

In § 2, by introducing the concept of partially ordered  $\varrho$ -set  $(X(\trianglelefteq); \{\phi_x\})$ , we construct a fundamental generalized inverse  $\ast$ -semigroup  $T_{X(\trianglelefteq)}(\mathcal{M})$ . Also, we shall see that  $T_{X(\trianglelefteq)}(\mathcal{M})$  has similar properties with  $T_{X(\pi)}(\mathcal{M})$ , where  $T_{X(\pi)}(\mathcal{M})$  has been given by T. Imaoka, I. Inata and H. Yokoyama [4]. And we shall show that two concepts, strong  $\pi$ -groupoids and partially ordered  $\varrho$ -sets, are equivalent.

In § 3, we shall introduce the notion of  $\omega$ -set  $(X(\preceq); \sigma)$ , and construct a generalized inverse  $\ast$ -semigroup  $T_{(X(\preceq); \sigma)}(\mathcal{M})$ . Furthermore, let  $S$  be a generalized inverse  $\ast$ -semigroup with the set of projections  $P$ , we shall make two generalized inverse  $\ast$ -semigroups  $T_{P(\trianglelefteq)}(\mathcal{M})$  and  $T_{(S(\preceq); \Omega)}(\mathcal{M})$ , where the former is obtained in § 2, and the latter is constructed in this section. Then we shall show that these three semigroups make a commutative diagram.

## 2 Fundamental generalized inverse $*$ -semigroups

### 2.1 $T_{X(\trianglelefteq)}(\mathcal{M})$

Let  $X(\trianglelefteq)$  be a partially ordered set and, for each  $x \in X$ , consider an order-preserving mapping  $\phi_x : X \rightarrow X$ . If a relation  $\rho = \{(x, y) \in X \times X : y\phi_x = x, x\phi_y = y\}$  is an equivalence relation on  $X$  such that

- (P1)  $x \trianglelefteq y \implies$  for each  $y' \in y\rho$ , there exists  $x' \in x\rho$  such that  $x' \trianglelefteq y'$ ,
- (P2) a relation  $\leq = \{(x\rho, y\rho) \in X/\rho \times X/\rho : \text{there exists } x' \in x\rho \text{ such that } x' \trianglelefteq y'\}$  is a partial order and  $X/\rho(\leq)$  is a semilattice,
- (P3)  $x_1 \trianglelefteq y, x_2 \trianglelefteq y$  and  $x_1\rho \leq x_2\rho \implies x_1 \trianglelefteq x_2$ ,

then  $(X(\trianglelefteq); \{\phi_x\})$  is called a *partially ordered  $\rho$ -set*.

Let  $(X(\trianglelefteq); \{\phi_x\})$  be a partially ordered  $\rho$ -set. Define an equivalence relation  $\mathcal{U}$  on  $\mathcal{X}$  by

$$\mathcal{U} = \{(\langle a \rangle, \langle b \rangle) \in \mathcal{X} \times \mathcal{X} : \langle a \rangle \simeq \langle b \rangle (\text{order isomorphic})\},$$

where  $\mathcal{X}$  is the set of all principal ideals of  $(X(\trianglelefteq); \{\phi_x\})$ . For  $(\langle a \rangle, \langle b \rangle) \in \mathcal{U}$ , let  $T_{\langle a \rangle, \langle b \rangle}$  be the set of all (order) isomorphisms of  $\langle a \rangle$  onto  $\langle b \rangle$ , and let

$$T_{X(\trianglelefteq)} = \bigcup_{(\langle a \rangle, \langle b \rangle) \in \mathcal{U}} T_{\langle a \rangle, \langle b \rangle}.$$

For any  $\alpha, \beta \in T_{X(\trianglelefteq)}$ , define a mapping  $\theta_{\alpha, \beta}$  as follows:

$$\theta_{\alpha, \beta} = \{(x, y) \in r(\alpha) \times d(\beta) : (x, y) \in \rho\},$$

where  $\rho$  is defined in  $(X(\trianglelefteq); \{\phi_x\})$ .

Then  $\theta_{\alpha, \beta} \in T_{X(\trianglelefteq)}$ . Let  $\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in T_{X(\trianglelefteq)}\}$ , and define a multiplication  $\circ$  and a unary operation  $*$  on  $T_{X(\trianglelefteq)}$  by

$$\begin{aligned} \alpha \circ \beta &= \alpha \theta_{\alpha, \beta} \beta, \\ \alpha^* &= \alpha^{-1}. \end{aligned}$$

Then it is clear that  $T_{X(\trianglelefteq)}(\circ, *)$  is a regular  $*$ -subsemigroup of the  $\iota$ -symmetric generalized inverse  $*$ -semigroup  $\mathcal{GI}_{(X; \rho)}(\mathcal{M})$ . Hence it is a generalized inverse  $*$ -semigroup and denoted by  $T_{X(\trianglelefteq)}(\mathcal{M})$ .

Let  $S$  be a generalized inverse  $*$ -semigroup and  $P = P(S)$ . We consider  $P$  as a partially ordered set with respect to the natural order. Now, we have the following results.

**Theorem 2.1** *A regular  $\ast$ -semigroup  $T_{X(\trianglelefteq)}(\mathcal{M})$  is a generalized inverse  $\ast$ -semigroup whose set of projections is order isomorphic to  $X(\trianglelefteq)$ .*

**Corollary 2.2** *A partially ordered set  $X$  is order isomorphic to the set of projections of a generalized inverse  $\ast$ -semigroup if and only if it is a partially ordered  $\rho$ -set.*

## 2.2 Representations

Let  $S$  be a generalized inverse  $\ast$ -semigroup. Hereafter, denote  $E(S)$  and  $P(S)$  simply by  $E$  and  $P$ , respectively. Let  $E \sim \sum\{E_i : i \in I\}$  be the structure decomposition of  $E$ , and let  $P_i = P(E_i)$ . For any  $e \in P$ , define a mapping  $\phi_e : P \rightarrow P$  by

$$f\phi_e = efe.$$

Let  $e, f \in P$ , define a relation  $\trianglelefteq$  on  $P$  by

$$e \trianglelefteq f \iff e = fef,$$

that is,  $\trianglelefteq$  is the restriction of natural order on  $S$  to  $P$ .

**Lemma 2.3** *The set  $(P(\trianglelefteq); \{\phi_e\})$ , defined above, is a partially ordered  $\rho$ -set.*

Now, we can consider the generalized inverse  $\ast$ -semigroup  $T_{P(\trianglelefteq)}(\mathcal{M})$ , where  $\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha \text{ and } \beta \text{ are order isomorphisms among principal ideals of } (P(\trianglelefteq); \{\phi_e\})\}$ .

**Lemma 2.4** *For any  $a \in S$ ,  $P(Sa)$  ( $= P(Sa^\ast a)$ ) is a principal ideal of  $(P(\trianglelefteq); \{\phi_e\})$ .*

For any  $a \in S$ , define a mapping  $\tau_a : \langle aa^\ast \rangle \rightarrow \langle a^\ast a \rangle$  by

$$e\tau_a = a^\ast ea,$$

where  $e \in \langle aa^\ast \rangle$ . It follows from [4] that  $\tau_a \in T_{S(\trianglelefteq)}$  and  $\tau_a^\ast = \tau_{a^\ast}$ . Moreover, for any  $a, b \in S$ ,  $\theta_{\tau_a, \tau_b} = \tau_{a^\ast abb^\ast}$ . And we have the following theorem.

**Theorem 2.5** *Let  $S$  be a generalized inverse  $\ast$ -semigroup such that  $E(S) = E$  and  $P(S) = P$ . Let  $E \sim \sum\{E_i : i \in I\}$  be the structure decomposition of  $E$  and  $P_i = P(E_i)$ . Denote the restriction of the natural order on  $S$  to  $P$  by  $\trianglelefteq$ . For any  $e \in P$ , define a mapping  $\phi_e : P \rightarrow P$  by  $f\phi_e = efe$ . Then  $(P(\trianglelefteq); \{\phi_e\})$  is a partially ordered  $\rho$ -set and  $T_{P(\trianglelefteq)}(\mathcal{M})$  is a generalized inverse  $\ast$ -semigroup.*

*Moreover, for any  $a \in S$ , define a mapping  $\tau_a : \langle aa^\ast \rangle \rightarrow \langle a^\ast a \rangle$  by  $e\tau_a = a^\ast ea$ . Then a mapping  $\tau : S \rightarrow T_{P(\trianglelefteq)}(\mathcal{M})$  ( $a \mapsto \tau_a$ ) is a  $\ast$ -homomorphism and the kernel of  $\tau$  is the maximum idempotent-separating  $\ast$ -congruence on  $S$ .*

Now, we have the following theorem.



**Theorem 2.6** *A generalized inverse  $*$ -semigroup  $S$  is fundamental if and only if it is  $*$ -isomorphic to a  $\mathcal{P}$ -full generalized inverse  $*$ -subsemigroup of  $T_{X(\trianglelefteq)}(\mathcal{M})$  on a partially ordered  $\varrho$ -set  $(X(\trianglelefteq); \{\phi_x\})$  such that  $P(T_{X(\trianglelefteq)}(\mathcal{M}))$  is order isomorphic to  $P(S)$ .*

Denote the sets of all partially ordered  $\varrho$ -sets and the set of all strong  $\pi$ -groupoids by  $\mathbb{P}$  and  $\mathbb{S}$ , respectively.

**Remark 2.7** *Let  $(X(\trianglelefteq); \{\phi_x\})$  be any element of  $\mathbb{P}$ . For any  $x\varrho, y\varrho \in X/\varrho$  ( $x\varrho \geq y\varrho$ ), define a mapping  $\overline{\varphi}_{x\varrho, y\varrho} : X_{x\varrho} \rightarrow X_{y\varrho}$  by*

$$x' \overline{\varphi}_{x\varrho, y\varrho} = y', \text{ where } y' \in y\varrho \text{ such that } y' \trianglelefteq x'.$$

Moreover, we define a partial product on  $X$  as follows:

$$xy = \begin{cases} x \overline{\varphi}_{x\varrho, (x\varrho)(y\varrho)} & \text{if } x \overline{\varphi}_{x\varrho, (x\varrho)(y\varrho)} = y \overline{\varphi}_{y\varrho, (x\varrho)(y\varrho)} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then  $(X(\trianglelefteq); \{\phi_x\})\lambda = X(\pi_\varrho; X/\varrho; \{\overline{\varphi}_{x\varrho, y\varrho}\})$  is a strong  $\pi$ -groupoid, where  $\pi_\varrho$  is the partition of  $X$  induced by  $\varrho$ .

Conversely, let  $X(\pi; Y; \{\varphi_{e,f}\})$  be any element of  $\mathbb{S}$ . For any  $x \in X$ , define a mapping  $\tilde{\phi}_x : X \rightarrow X$  by

$$y \tilde{\phi}_x = x \varphi_{e,ef},$$

where  $x \in X_e$  and  $y \in X_f$ . If we define  $\blacktriangleleft = \{(x, y) \in X \times X : x \tilde{\phi}_y = x\}$ , then  $X(\pi; Y; \{\varphi_{e,f}\})\mu = (X(\blacktriangleleft); \{\tilde{\phi}_x\})$  is a partially ordered  $\varrho$ -set.

Hence the mappings  $\lambda, \mu$  from  $\mathbb{P}$  to  $\mathbb{S}$  and from  $\mathbb{S}$  to  $\mathbb{P}$ , respectively, are well-defined. Moreover  $\mu\lambda = 1_{\mathbb{S}}$ , and for any  $(X(\trianglelefteq); \{\phi_x\}) \in \mathbb{P}$ , if  $(X(\trianglelefteq); \{\phi_x\})\lambda\mu = (X(\blacktriangleleft); \{\tilde{\phi}_x\})$ , then  $\trianglelefteq = \blacktriangleleft$ .

By the above argument, for any  $(X(\trianglelefteq); \{\phi_x\})$  in  $\mathbb{P}$ , without loss of generality, we can consider  $(X(\trianglelefteq); \{\phi_x\})$  as a member of  $\mathbb{P}\lambda\mu$ .

Now, let  $X(\pi; Y; \{\varphi_{e,f}\})$  be any element of  $\mathbb{S}$ . If  $X(\pi; Y; \{\varphi_{e,f}\})\mu = (X(\trianglelefteq); \{\phi_x\})$ . Then we can construct two generalized inverse  $*$ -semigroups  $T_{X(\pi)}(\mathcal{M})$  and  $T_{X(\trianglelefteq)}(\mathcal{M})$ . In this case, these two generalized inverse  $*$ -semigroups are  $*$ -isomorphic.

### 3 Extensions of $T_{X(\trianglelefteq)}(\mathcal{M})$

#### 3.1 $T_{(X(\preceq);\sigma)}(\mathcal{M})$

By a *pre-order* on a set  $X$  we shall mean a reflexive and transitive relation. Let  $X(\preceq)$  be a pre-ordered set and let  $\nu = \{(a, b) \in X \times X : a \preceq b \text{ and } b \preceq a\}$ . Then  $\nu$  is an equivalence relation on  $X$  and  $X/\nu$  is a partially ordered set with respect to the induced relation

$$(C1) \quad a\nu \trianglelefteq b\nu \text{ if and only if } a \preceq b.$$

We call  $\trianglelefteq$  the *naturally induced order* on  $X/\nu$  from  $\preceq$ . Clearly  $\nu$  is the smallest equivalence relation on  $X$  for which (C1) defines a partial order on  $X/\nu$ . We call  $\nu$  the *minimum partial order congruence* (mpo-congruence) on  $X$  from  $\preceq$ .

A subset  $A$  of  $X$  is an *ideal* of  $X$  provided that  $x \preceq y$  and  $y \in A$  implies  $x \in A$ . For  $a \in X$ , we call  $\{x \in X : x \preceq a\}$  the *principal ideal generated* by  $a$  and denote it by  $\langle a \rangle$ .

A bijection  $\alpha$  of one pre-ordered set  $X$  onto another  $Y$  will be called an *isomorphism* provided that, for  $a, b \in X$ ,  $a \preceq b$  if and only if  $a\alpha \preceq b\alpha$ . In particular, if  $\nu_X$  and  $\nu_Y$  denote the respective mpo-congruences then  $(a, b) \in \nu_X$  if and only if  $(a\alpha, b\alpha) \in \nu_Y$ .

Let  $X(\preceq)$  be a pre-ordered set and  $\nu$  the mpo-congruence from  $\preceq$ . Then  $X$  is a *partially pre-ordered  $\varrho$ -set* if and only if  $X/\nu$  is a partially ordered  $\varrho$ -set with respect to the naturally induced order  $\trianglelefteq$  from  $\preceq$ .

Let  $X(\preceq)$  be a partially pre-ordered  $\varrho$ -set and  $\sigma$  an equivalence relation on  $X$  such that

- (O1) for any  $x$  in  $X$ ,  $\langle x \rangle$  is an  $\iota$ -single subset with respect to  $\sigma$ ,
- (O2) for  $x, y$  in  $X$ , if  $(x, y) \in \sigma$  then  $(x\nu, y\nu) \in \varrho$ ,
- (O3) for  $x, y, z$  in  $X$ , if  $(x\nu)\varrho \wedge (y\nu)\varrho = (z\nu)\varrho$ ,  $z_1\nu \trianglelefteq x\nu$  and  $z_2\nu \trianglelefteq y\nu$  ( $z_1\nu, z_2\nu \in (z\nu)\varrho$ ), then for any  $a \in \langle z_i \rangle$ , there exists  $b \in \langle z_j \rangle$  such that  $(a, b) \in \sigma$ , where  $1 \leq i, j \leq 2$ .

Then  $(X(\preceq); \sigma)$  is called an  $\omega$ -set.

Let  $(X(\preceq); \sigma)$  be an  $\omega$ -set and let  $T_{(X(\preceq);\sigma)}$  denote the set of all isomorphisms from a principal ideal onto another one.

For any  $\alpha, \beta \in T_{(X(\preceq);\sigma)}$ , define a mapping  $\theta_{\alpha, \beta}$  as follows:

$$\theta_{\alpha, \beta} = \{(a, b) \in r(\alpha) \times d(\beta) : (a, b) \in \sigma\}.$$

Then  $\theta_{\alpha,\beta} \in T_{(X(\preccurlyeq);\sigma)}$ . Let  $\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha, \beta \in T_{(X(\preccurlyeq);\sigma)}\}$ , and denote a multiplication  $\circ$  and a unary operation  $*$  on  $T_{(X(\preccurlyeq);\sigma)}$  by

$$\begin{aligned}\alpha \circ \beta &= \alpha \theta_{\alpha,\beta} \beta, \\ \alpha^* &= \alpha^{-1}.\end{aligned}$$

Clearly,  $\alpha \circ \beta$  is an isomorphism from  $\langle z_1 \alpha^{-1} \rangle$  onto  $\langle z_2 \beta \rangle$ . It is obvious that  $T_{(X(\preccurlyeq);\sigma)}(\circ, *)$  is a regular  $*$ -semigroup. Hence it is a generalized inverse  $*$ -semigroup and denoted by  $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$ .

**Theorem 3.1** *A regular  $*$ -semigroup  $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$  is a generalized inverse  $*$ -subsemigroup of  $\mathcal{GI}_{(X;\sigma)}(\mathcal{M})$  whose set of projections is order isomorphic to  $X/\nu$ .*

**Remark 3.2** *In  $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$ , if  $\preccurlyeq = \trianglelefteq$  and  $\sigma = \varrho$  then  $T_{(X(\trianglelefteq);\varrho)}(\mathcal{M}) = T_{X(\trianglelefteq)}(\mathcal{M})$ .*

Let  $(X(\preccurlyeq);\sigma)$  be an  $\omega$ -set and let  $Y = X/\nu$ , where  $\nu$  is the mpo-congruence from  $\preccurlyeq$ . For any element  $\alpha$  in  $T_{(X(\preccurlyeq);\sigma)}$ , assume that  $d(\alpha) = \langle a \rangle$ . Then we can define a new mapping  $\alpha' \in T_{Y(\trianglelefteq)}$  as follows:

$$\begin{aligned}d(\alpha') &= \{x\nu : x \in d(\alpha)\}, \\ (x\nu)\alpha' &= (x\alpha)\nu.\end{aligned}$$

Then  $\alpha' \in T_{Y(\trianglelefteq)}$ . Now, define a mapping  $\xi : T_{(X(\preccurlyeq);\sigma)}(\mathcal{M}) \rightarrow T_{Y(\trianglelefteq)}(\mathcal{M})$  by  $\alpha\xi = \alpha'$ . Then, it is easy to see that  $\xi$  is a  $*$ -homomorphism.

**Proposition 3.3** *The mapping  $\xi : \alpha \mapsto \alpha'$  of  $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$  into  $T_{Y(\trianglelefteq)}(\mathcal{M})$  is a  $*$ -homomorphism of  $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$  onto a  $\mathcal{P}$ -full generalized inverse  $*$ -subsemigroup of  $T_{Y(\trianglelefteq)}(\mathcal{M})$  such that  $\xi \circ \xi^{-1} = \mu$ , where  $\mu$  is the maximum idempotent separating  $*$ -congruence on  $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$ .*

Hereafter, we shall refer to  $\xi$  as the *natural projection* of  $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$  to  $T_{Y(\trianglelefteq)}(\mathcal{M})$ .

### 3.2 Inflated representations

Let  $S$  be a generalized inverse  $*$ -semigroup. Hereafter, denote  $E(S)$  and  $P(S)$  simply by  $E$  and  $P$ , respectively. Define a relation  $\preccurlyeq$  on  $S$  by:

$$a \preccurlyeq b \text{ if and only if } a^*a \leq b^*b,$$

for  $a, b \in S$ . Then clearly  $\preccurlyeq$  is a pre-order on  $S$  for which the mpo-congruence from  $\preccurlyeq$  is  $\nu = \mathcal{L}$ . Hence  $S/\mathcal{L} = S/\nu$ , under the naturally induced order  $\trianglelefteq$  from  $\preccurlyeq$ , is just the set of  $\mathcal{L}$ -classes of  $S$  under the usual partial ordering of the  $\mathcal{L}$ -classes of a generalized inverse  $*$ -semigroup and so is order isomorphic to the partially ordered  $\varrho$ -set  $P$  of  $S$ . Hence  $S$  is a partially pre-ordered  $\varrho$ -set under  $\preccurlyeq$ . Then  $\varrho = \mathcal{J}^E|_P$  and hence  $(a\nu)\varrho(b\nu) \iff a^*a\mathcal{J}^Eb^*b$ . Hereafter, for any  $a \in S$ , we think  $a\nu = L_{a^*a}$  as  $a^*a$ .

For any  $a \in S$ , define a mapping  $\rho_a : Sa^* \rightarrow Sa$  as follows:

$$\begin{aligned} d(\rho_a) &= Sa^*(= Saa^*), \\ x\rho_a &= xa. \end{aligned}$$

Let  $\rho : S \rightarrow \mathcal{GI}_{(S;\Omega)}(\mathcal{M})$  by  $a\rho = \rho_a$ , where the relation  $\Omega$  defined by: for  $x, y \in S$ ,

$$(x, y) \in \Omega \iff x\rho_e = y \text{ for some } e \in E.$$

Since  $S$  is a regular  $*$ -semigroup, the representation  $\rho$  is faithful. Moreover, it follows from [6, Lemma 3.3] that it is a  $*$ -monomorphism.

**Lemma 3.4** *The set  $(S(\preccurlyeq); \Omega)$ , defined above, is an  $\omega$ -set.*

Again, we consider  $\rho_a : Sa^* \rightarrow Sa$ . By Lemma 3.4,  $d(\rho_a) = \langle a^* \rangle$  and  $r(\rho_a) = \langle a \rangle$ . For  $x, y \in d(\rho_a)$ ,  $x^*x, y^*y \leq a^*a$ . Now  $x \preccurlyeq y$  if and only if  $x^*x \leq y^*y$  while  $xa \preccurlyeq ya$  if and only if  $a^*x^*xa = (xa)^*(xa) \leq (ya)^*(ya) = a^*y^*ya$ . But, since  $x^*x, y^*y \leq a^*a$  it follows that  $x^*x \leq y^*y$  if and only if  $a^*x^*xa \leq a^*y^*ya$ . Therefore  $x \preccurlyeq y$  if and only if  $xa \preccurlyeq ya$ . Thus  $\rho_a$  is an isomorphism of  $\langle a^* \rangle$  onto  $\langle a \rangle$ , and hence  $S\rho \subseteq T_{(S(\preccurlyeq); \Omega)}(\mathcal{M})$ .

Now, we have the following theorem.

**Theorem 3.5** *Let  $S$  be a generalized inverse  $*$ -semigroup and define the relation  $\preccurlyeq$  on  $S$  by  $a \preccurlyeq b$  if and only if  $a^*a \leq b^*b$ . Then  $\preccurlyeq$  is a pre-order on  $S$  with respect to which  $S$  is a partially pre-ordered  $\varrho$ -set, moreover  $(S(\preccurlyeq); \Omega)$  is an  $\omega$ -set. The faithful representation  $\rho$  of  $S$  embeds  $S$  as a  $\mathcal{P}$ -full generalized inverse  $*$ -subsemigroup of  $T_{(S(\preccurlyeq); \Omega)}(\mathcal{M})$ .*

*If  $\nu$  is the mpo-congruence on  $S$  from  $\preccurlyeq$ , then  $\nu = \mathcal{L}$  and  $S/\nu$  is order isomorphic to the partially ordered  $\varrho$ -set  $P$  of  $S$ . Moreover,  $\rho\xi = \tau$ , where  $\xi$  is the natural projection and  $\tau$  is the representation which is defined in Theorem 2.5.*

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